Seweryn Blandzi, Stanisław Wałcerz
(Warszawa) (Poznań)

BUILDING REGULAR PATTERNS WITH SIDE, DIAGONAL AND POLYGONAL NUMBERS*

1. THE ORIGIN OF THE MODEL

The models presented in this paper have been known for more than two thousand years and originate from the ancient Grecian mathematicians. It is worthwhile to remember that the ancient philosophers studied properties of numbers, as the numbers had been attributed with divine significance, so the rules governing the numbers were assumed to be the same as those governing every process of the Universe. In the Pythagorean-Platonic tradition the unity (*monas, hen – metron*) played a distinguished role as a principle of formation and ontological-epistemological *prius*. Second principle was an indeterminate duality (*aoristos dyas*), the principle of multiplication and division. Both factors were recognized as the ultimate principles of all beings (*eidetic numbers*, ideas, numbers, geometrical and physical bodies). A natural number had not only been identified with the power of a set of elements (*systema monadon*), a number as such was conceived as a real power of the Nature. (resp. metaphysical substance) as well as it expressed pure ontological relations (*logoi*). In the Pythagorean philosophy the numbers also helped in symbolic understanding of the world. For example the number 5 symbolised properties of physical bodies including their colours (1 – a point, 2 – a line, 3 – a flat figure, 4 – a body, 6 – life, 7 – mind, 8 – love, 9 – reason), whereas the number 10 was treated as the symbol of perfection. Aristotle says that 10 contains the whole nature of the numbers [1], so 10 was called the 'Arch – four'

* The talk was presented at the International Conference „Mathematics and Art", Moscow (Suzdal), September 1996.

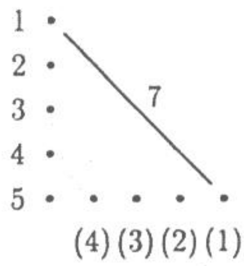

Fig. 1. The side number versus the diagonal number.

(*tetraktys*) and it was presented in the form of the triangle constructed of 10 points. Methods of graphical representation of certein properties of the natural numbers were improved in order to study such the phenomena as proportion, symmetry and commensurability. It is then clear that only the numbers known now as the natural and rational ones were the subject of interest (see for example [2]). It was Isaac Newton who renewed and promoted a meaning of the number as being the *measure* and *ratio* of quantities [3].

2. THE CONCEPT OF SIDE AND DIAGONAL NUMBERS

The system of numbers known as the *side* and *diagonal* ones received much attention from the ancient mathematicians [2, 4]. The description of this system comes from Theon of Smyrna. The unit (monad), as the beginning of all entities, originates both a *side* and a *diameter*. So the monad was the first side number as well as the first diagonal one. Denoting side numbers as a_n and diagonal numbers as d_n one may write

$$a_1 = 1, \qquad \mathrm{d}_1 = 1,$$
$$a_{n+1} = a_n + d_n, \qquad d_{n+1} = 2a_n + d_n \,. \tag{1}$$

Theon states the proposition that

$$d_n^2 = 2a_n^2 + (-1)^n \tag{2}$$

so the sum of squares of *all* diagonal numbers is the doubled sum of squares of *all* side numbers. It is seen that the succesive fractions of $d_n \,/\, a_n$ approximate better and better the value of $\sqrt{2}$: $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$

The side and diagonal numbers can be presented in a diagram like that in Fig. 1. As follows, the side number counted at the double course race gives its square (cf. [5]). However, the diagonal itself could be expressed by neither side nor diagonal numbers, so Plato [6] pointed out the contrast between $\sqrt{50}$ as the *irrational* (Phytagorean) diameter of 5 and the *rational* diameter equal to the approximation

$\sqrt{50-1} = 7$. Notice that $d_n = 7$ is the diagonal number corresponding to the side number $a_n = 5$.

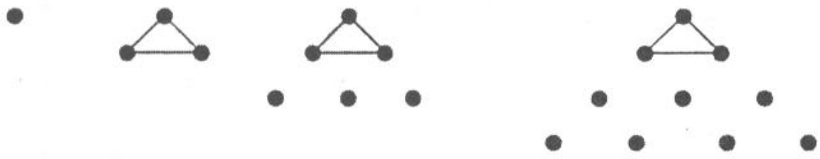

Fig. 2. The first triangular numbers: 1, 3, 6 and 10. The second tringular number ($c_2(3) = 3$) is marked.

3. THE EQUILATERAL POLYGONS AS REPRESENTATIONS OF THE NUMBERS

The ancient philosophers used to classify the numbers according to series of equilateral polygons that could represent certain sets of separate items (points, stones etc.). The idea of constructing the appropriate polygons is as follows:

1. The series consists of polygons of the same number of sides, e.g. triangles (in general k-kind polygons – of k sides).
2. The first polygon of any kind consists of one item (as the monad was the origin of everything).
3. The second polygon of k-kind consists of just k items put at the vertexes of equilateral polygon of k sides, the length (in arbitrary units) of the side is equal 1: thus the distance between items (points) at the side has been fixed. Each side contains two items placed at the ends. The total number of items is equal k. From this step the item, put as the first one, will always be the vertex of plygons built at next steps.
4. Every next polygon contains the polygons built so far. The third polygon has sides of length equal to 2 units and 3 items at the each side: two items at the ends and one item in the middle of the side. Moreover, two sides of the second polygon partially coincide with two sides of the third polygon. Each side of a polygon being built has to be parallel to the relevant side of the polygons built so far. These rules are valid for next pairs of polygons, respectively.
5. The subsequent polygons are obtained through incresing the length of the side by one unit.

The whole procedure is illustrated in Fig. 2, which shows that the numbers that the triangles can be built of are: 1, 3, 6, 10, 15, 21... Note that the triangular number represented by the triangle built at a given step is a sum of all items forming the

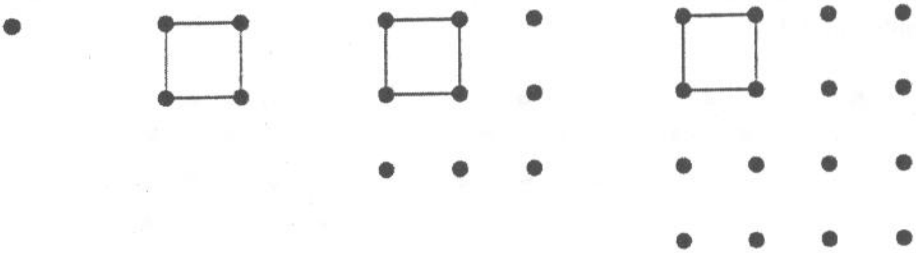

Fig. 3. The first square numbers: 1, 4, 9, 16. The second square number ($c_2(4) = 4$) is marked.

triangles built so far. The same procedure applied to square numbers is shown in Fig. 3.

4. THE DERIVATION OF THE POLYGONAL NUMBERS

The number of the kind k, obtained at the n-th step will be written down as $c_n(k)$. It is easy to determine the sequence of triangular numbers even without plotting the succeeding triangles: at the step number n one adds just the integer n:

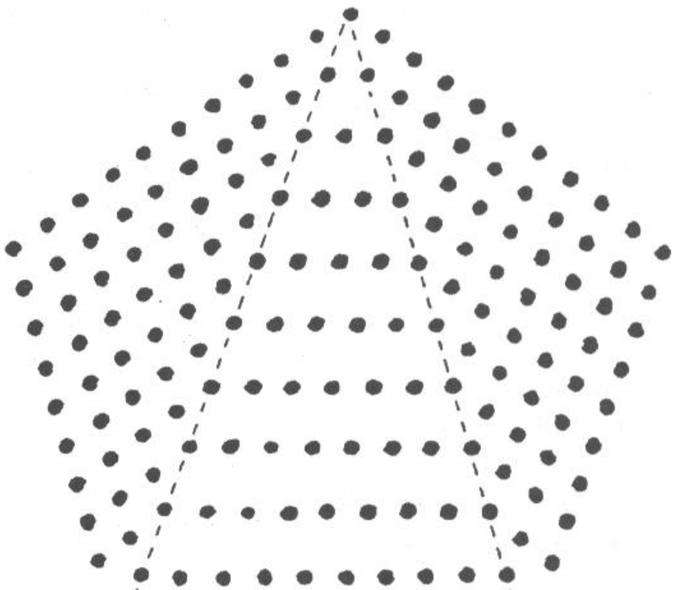

Fig. 4. The pentagonal numbers.

$c_1(3) = 1$
$c_2(3) = 1 + 2$
$c_3(3) = 1 + 2 + 3$
$c_4(3) = 1 + 2 + 3 + 4$ and so on.

Derivation of the square number does not give any trouble: it is enough to multiply n times n in order to obtain the relevant square number, so $\forall_n\ c_n(4) = n^2$. However, the difficulty arises in the case of polygonal numbers different than the triangular or square ones. Grecian mathematicians built up a table of sequentially ordered triangle, square, pentagonal, hexagonal... numbers, that let them derive any number on the basis of formerly written numbers. Since this procedure is not essential for the purpose of this paper, it will not be presented here. Moreover, it is possible now to determine any number $c_n(k)$ of any kind k, according to certain recursive formulas, e.g.:

$$c_n(k) = c_{n-1}(k) + (n-2) \times (k-2) + (k-1) \qquad (3)$$

where k is the number of sides (and vertexes) of the polygon. For $n > 2$ each side of a polygon includes $n-2$ side points between vertexes. The interpretation of the formula (3) is as follows: the polygonal number $c_n(k)$ built at n-th step consists of:

1. the former number $c_{n-1}(k)$,
2. plus (the number $n-2$ of side points) times ($k-2$ sides),
3. plus $k-1$ vertexes. One vertex remains common with the polygons built so far.

In order to obtain the proper values of k-kind numbers it is necessary to put

$$c_0(k) = 0, \quad c_1(k) = 1 \qquad (4)$$

irrespective of the number n. The reason why the ancient mathematicians did not know the formulas like (3) is that they were not able to write down the term c_0, as they did not use the digit zero.

5. THE PENTAGONAL NUMBERS AS A MODEL OF THE PHYSICAL STRUCTURE

It has been shown that the series of polygons could represent certain numbers. Henceforth, the terms *polygonal numbers* and *polygons* can be used equivalently unless it does not lead to misunderstandings.

Let us construct the series of pentagonal numbers. According to the formula (3) they are: 1, 5, 12, 22, 35, 51, 70... (zero has been omitted). The graph presenting the series of pentagonal numbers up to 176 is shown at Fig. 4.

It might be interesting to notice that the points at the graph not only form the succeeding pentagons. There are visible 3 areas: one in the middle and two at the sides of the graph. Within each area the periodic order of Bravais type is preserved. The middle area has edges (marked with dotted lines) coinciding with the edges of the side areas. The cincidence of the edges is perfect, i.e. the domains do not need to be adjusted to each other in any additional way, there are no defects, vacancies or dislocations. In other words the pentagonal numbers form a regular pattern consisting of three crystal-like domains strictly fitted to each other. Watching this areas separately one can say that they are three crystallographic domains in two dimensions. However, as a whole the stucture does not present a model of a crystal belonging to any of the Bravais class [7]. Nevertheless, the structure of pentagonal numbers type produces the diffraction of light, with the occurrence of the ten-fold Bragg diffraction pattern, what is a characteristic feature of the so-called quasicrystals. This a little bit surprising effect has been observed experimentally with the help of the diffraction grating prepared as a micro-slide [8], both positive and negative, of the pentagonal number. The ten-fold pattern occurs even in a visible light obtained from a small electric bulb (that can be regarded as a point source of light). In a sense, the pentagonal numbers suggest another model of the quasicrystalline structure, described with the Penrose tilling.

References

[1] Aristotle: *Metaphysics* 986 a 8 sq.

[2] Euclid: *The Thirteen Book of the Elements.*

[3] I. Newton: *Arithmetica Universalis.*

[4] D.H. Fowler: "Bull. Amer. Math. Soc." T. 1 (1979) p. 807.

[5] Nikomachos of Gerasa: *Introductio arithmetica.*

[6] Plato: *Republic* (546 C).

[7] A. Bravais: "J. Ecole Polytech." T. 19 (1850) p. 1.

[8] For the technique of preparation of such the slides see e.g.: W. Gorzkowski in *Symmetry and Structural Properties of Condensed Matter*. "Proc. of the Int. School of Theor. Phys.". World Sci. Singapore 1991.

Seweryn Blandzi, Stanisław Wałcerz

BUDOWANIE WZORÓW REGULARNYCH Z LICZB BRZEGOWYCH, PRZEKĄTNIOWYCH I WIELOKĄTNYCH

Poszukiwania pochodzenia i natury liczb, prowadzone w antycznej Grecji, spowodowały rozwój metod służących do geometrycznej reprezentacji zarówno samych liczb naturalnych, jak i zjawisk wyrażanych liczbowo: proporcji, współmierności i symetrii.

W tradycji platońskiej jedność (*monas, hen – metron*) odgrywała wyróżnioną rolę jako zasada tworzenia i ontologiczno-epistemologiczne *prius*. Liczba naturalna była nie tylko identyfikowana z mocą zbioru, to jest liczbą elementów (*systema monadon*); liczba jako taka traktowana była jako rzeczywista siła przyrody.

System liczb znany jako liczby brzegowe (boczne) i przekątniowe (diagonalne), którego opis pochodzi od Teona ze Smyrny, był obiektem szerokiego zainteresowania starożytnych matematyków. Jedność (monada), jako początek wszelkich bytów, jest zasadą zarówno brzegu, jak i przekątnej, a więc jedność ma być pierwszą liczbą brzegową (a_1) oraz pierwszą liczbą przekątniową (d_1). Teon stwierdził, że suma kwadratów wszystkich liczb przekątniowych d_n jest równa podwojonej sumie wszystkich liczb brzegowych a_n. Kolejne ilorazy d_n / a_n przybliżają coraz lepiej liczbę jednak, jak zauważył Platon, rzeczywista (pitagorejska) przekątna nie jest dokładnie równa odpowiedniej liczbie przekątniowej.

Klasyfikacja liczb naturalnych poprzez przypisanie im odpowiednich wielokątów jest pierwszym przykładem badania zjawisk symetrii w sposób sformalizowany. Liczbę klasyfikowano jako n – kątną, jeżeli z n punktów można w odpowiedni sposób skonstruować n-kąt foremny, na przykład liczby trójkątne to: 1, 3, 6, 10, 15... (rys. 2), liczby czworokątne to: 1, 4, 9, 16... (rys. 3), liczby pięciokątne to: 1, 5, 12, 22, 35... (rys. 4). Jedność, jako zasada wszystkiego, jest początkową liczbą każdego rodzaju, a każda liczba n-kątna zawiera w sobie wszystkie poprzednie liczby o tej samej symetrii n-krotnej. Grecy zbudowali tablice kolejnych liczb trój-, cztero-, pięcio- itd. kątnych, co umożliwiało im znalezienie dowolnie dużej liczby dowolnego rodzaju. Obecnie można podać wyrażenie ogólne na k-tą liczbę n-kątną (3), przyjmując $c_0 = 0$, $c_1 = 1$ dla dowolnego n. Ponieważ Grecy nie zapisywali liczby zero, wyrażenie typu (3) było nieznane.

Obecnie interesujące jest to, że liczby wielokątne mogą mieć, jak się wydaje, zastosowanie w fizyce ciała stałego. Liczba pięciokątną opisuje strukturę regularną, zbudowaną z idealnie dopasowanych trzech domen o strukturze krystalicznej, jednak jako całość nie jest modelem kryształu, należącego do jednej z klas Bravais. Niemniej, struktury takie jak liczby pięciokątne powodują dyfrakcję przechodzącego promieniowania, charakteryzującą się obecnością dziesięciokrotnie złożonego wzoru Bragga.